Time-Dependent Solutions to the Fokker-Planck Equation of Maximum Reduced Air-Sea Coupling Climate Model

FENG Guolin, DONG Wenjie and CAO Hongxing

1Laboratory for Climate Studies of China Meteorological Administration, National Climate Center, Beijing 100081
2Key Laboratory of Regional Climate-Environment Research for Temperate East Asia (RCE-TEA), Chinese Academy of Sciences, Beijing 100029
3Chinese Academy of Meteorological Sciences, Beijing 100081

(Received May 13, 2005; revised September 14, 2005)

ABSTRACT

The time-dependent solution of reduced air-sea coupling stochastic-dynamic model is accurately obtained by using the Fokker-Planck equation and the quantum mechanical method. The analysis of the time-dependent solution suggests that when the climate system is in the ground state, the behavior of the system appears to be a Brownian movement, thus reasoning the foothold of Hasselmann’s stochastic climatic model; when the system is in the first excitation state, the motion of the system exhibits a form of time-decaying, or under certain condition a periodic oscillation with the main period being 2.3 yr. At last, the results are used to discuss the impact of the doubling of carbon dioxide on climate.

Key words: probability density, air-sea coupling stochastic-dynamic model, eigenfunction solution, greenhouse effect, climate model, Fokker-Planck equation

1. Introduction

Air-sea interaction is an important physical process in the climate system. Because oceans occupy two thirds of the earth’s surface, and have a tremendous thermal inertia, oceans exert an extremely important influence on atmospheric motion, and the air-sea interaction becomes a core item of climate change studies. Contrarily, the atmosphere constrains the motion of seawater through wind drifts and heat transfer. With regard to the hotspot problem of global warming, the ocean is a most important reservoir for carbon dioxide, and it absorbs about $1 \times 10^{12}-3 \times 10^{12}$ kg of carbon from the atmospheric carbon dioxide each year, which accounts for 40%-50% of the total carbon dioxide emission from fossil fuel burning. Therefore, scientists not only analyzed observational facts to reveal the coupling between atmosphere and ocean, but also developed air-sea coupling models of different complex extent to investigate the physical process. Li and Huang (1984) derived a two-variable linear coupling model from the parameterization of diabatic process, and discussed the spectral characteristics of the stochastic-dynamical model. Feng et al. (2001) discussed the approximate solution of the model, and introduced the effect of carbon dioxide into the model. In this paper, we apply the eigenfunction solution principle in quantum mechanics, yielding the time-dependent solution of the model. Then the analytical solution is used to investigate the greenhouse effect of carbon dioxide doubling and the change in the behavior of the model, yielding some meaningful results.

2. The steady state solution to the system of equations

The air-sea coupling stochastic-dynamic model equations (see Li and Huang, 1984; Feng et al., 2001) can be written as

\[
\frac{dT}{dt} = -\alpha_1 T + \beta_1 T_s + \omega_1(t),
\]

(1)

\[
\frac{dT_s}{dt} = -\alpha_2 T_s + \beta_2 T + \omega_2(t),
\]

(2)
where $T = \bar{T} - \bar{T}$, $T_s = \bar{T}_s - \bar{T}_s$ are the deviations of air temperature $\bar{T}$, and sea surface temperature $\bar{T}_s$, from the equilibrium state $\bar{T}$ and $\bar{T}_s$, respectively. The variables $\alpha_1, \alpha_2$ and $\beta_1, \beta_2$ are self-feedback coefficients and mutual-feedback coefficients, respectively, and they are associated with the solar constant, effective emittance ($\varepsilon$), specific heat at constant pressure, densities of air and water, the exchange coefficient of sensible heat flux, etc., see Li and Huang (1984) and Feng et al. (2001) for details. Carbon dioxide concentration is introduced into the model through its concentration is introduced into the model through its effective emittance ($\varepsilon_c = a\ln[\text{CO}_2] + b$, that is to say, $\varepsilon = \varepsilon_c + \varepsilon^*$, where $\varepsilon^*$ is the effective emittance of all gases except CO$_2$. The variables $a$ and $b$ are empirical coefficients ($a=0.0235$, $b=0.0537$); $\omega_1(t)$ and $\omega_2(t)$ are stochastic noises, and both assumed to be the white noises with zero mean value for convenience, i.e.,

$$
\langle \omega_1(t) \rangle = \langle \omega_2(t) \rangle = 0,
\langle \omega_1(t)\omega_1(t') \rangle = 2\gamma\delta(t-t'),
\langle \omega_2(t)\omega_2(t') \rangle = 2\gamma\delta(t-t'),
$$

(3)

where $\gamma$ is the intensity of noise, and $\delta$ is the Dirac function. Differentiating Eq.(2) with respect to $t$ and eliminating $T$ and $dT/dt$ by using Eq.(1) yield

$$
d^2T_s/dt^2 + \gamma dT_s/dt + \omega_0^2T_s = \omega(t),
$$

(4)

where $\gamma = \alpha_1 + \alpha_2$, $\omega_0^2 = \alpha_1\alpha_2 - \beta_1\beta_2 \geq 0$, and $\omega(t) = \alpha_1\omega_2(t) + \beta_2\omega_1(t)$.

### 2.1 Fokker-Planck equation

In order to accurately solve Eq.(4), it is transformed into a Langevin equation group:

$$
\dot{T}_s = v,
\dot{v} = -\gamma v - f'(T_s) + \omega(t),
$$

(5)

where $f'(T_s) = \omega_0^2T_s$. Then Eq.(5) may be written in the form of Fokker-Planck equation (see Rishen, 1984; Feng et al., 2002):

$$\frac{\partial W}{\partial t} = L_k W$$

$$L_k = L_k(T_s, v) = -\frac{\partial}{\partial T_s}v + \left(\gamma v + f'(T_s)\right)$$

+ $q\frac{\partial^2}{\partial T_s^2}$.

(6)

where $W$ is the probability density distribution function of $T$, $v$, $t$, and $L$ is the Fokker-Planck equation operator. $f(T_s)$ is a potential function, and $f'(T_s)$ is its derivative. Equation (6) can be further written in the form of the continuity equation of probability flow

$$\frac{\partial W}{\partial t} + \frac{\partial S_{T_s}}{\partial T_s} + \frac{\partial S_v}{\partial v} = 0,$n

(7)

where

$$S_{T_s} = T_s W,$

$$S_v = -\left[\gamma v + f'(T_s)\right]W - q\frac{\partial W}{\partial T_s}.$n

(8)

### 2.2 Steady state solution

The steady state solution can be obtained if and only if probability flows $S_{T_s}$ and $S_v$ are equal to constants everywhere of $T_s, v$, i.e., they do not change with time. As is known from Eq.(8), if $\partial S_v/\partial v$ wanted, Eq.(9) should be valid,

$$q\frac{\partial W}{\partial T_s} = -[\gamma v + f'(T_s)]W.$n

(9)

Then, distribution function $W(T_s, v)$ is obtained through integrating Eq.(7) with respect to time,

$$W(T_s, v) = A \cdot \exp \left[-\frac{\gamma}{2q}(v^2 + \omega_0^2T_s^2)\right],$$

(10)

where $A$ is determined by normalizing condition, i.e.,

$$\int W(T_s, v)dT_s dv = 1,$n

(11)

thus the same result as in the paper by Feng et al. (2001) is derived from different ways.

### 3. Time-dependent solution

According to the definition of first order moment in probability theory, the mean value of $T_s$ can be calculated from Eq.(10),

$$\langle T_s \rangle = \int T_s W(T_s, v)dT_s dv.$n

(12)

However, the temporal change of $\langle T_s \rangle$ is still unknown, therefore it is necessary to get the unsteady state solution (time-dependent solution) of Eq.(6). The Fokker-Planck equation operator in Eq.(6) can be written as
the sum of a reversible operator and an irreversible one, i.e.,
\[
L_k = L_{rev} + L_{ir},
\]
\[
L_{rev} = -v \frac{\partial}{\partial T_s} + f'(T_s) \frac{\partial}{\partial v},
\]
\[
L_{ir} = \gamma \frac{\partial}{\partial v}(v + \frac{q}{\gamma} \frac{\partial}{\partial v}).
\]

Similarly, a suitable transform of \( \tilde{L}_{rev} \) yields
\[
\tilde{L}_{rev} = -v \frac{\partial}{\partial T_s} + f'(T_s) \frac{\partial}{\partial v} - \frac{1}{2} f''(T_s) \frac{v \gamma}{q}
= L_{rev} - \frac{1}{2} f''(T_s) \frac{v \gamma}{q} = -bD - b^+ D,
\]
where
\[
D = \sqrt{\frac{q}{\gamma}} \frac{\partial}{\partial T_s}, \quad \tilde{D} = \sqrt{\frac{q}{\gamma}} \frac{\partial}{\partial T_s} + f'(T_s) \sqrt{\frac{q}{\gamma}}.
\]

Thus, the Fokker-Planck equation operator may be written as
\[
\tilde{L}_k = \tilde{L}_{rev} + \tilde{L}_{ir}
= -\gamma b^+ b - bD - b^+ D.
\]

Since \( f(T_s) \) does not explicitly contain time \( t \), distribution function \( W(T_s, v, t) \) may be written as
\[
W(T_s, v, t) = W_\lambda(T_s, v) \cdot f_\lambda(t).
\]

In like manner, the eigenvalue and corresponding eigenfunction of Fokker-Planck equation operator are
\[
\varphi(T_s, v, t) = \varphi(T_s, v) e^{-\lambda t},
\]
\[
L_k \varphi(T_s, v) = -\lambda \varphi(T_s, v),
\]
and their transformed forms are
\[
\tilde{L}_k \tilde{\varphi}(T_s, v) = -\lambda \tilde{\varphi}(T_s, v),
\]
\[
\tilde{\varphi}(T_s, v) = \exp(\frac{\gamma}{4q} (v^2 + f(T_s))) \varphi(T_s, v).
\]

Since the commutation of \( D \) and \( \tilde{D} \) is \( \omega_0^2 \), we define the Boson-operator as
\[
\alpha = \frac{\tilde{D}}{\omega_0} = \sqrt{\frac{2}{\omega_0}} \frac{\partial}{\partial T_s} + \frac{\omega_0}{2 \sqrt{\gamma}} T_s,
\]
\[
\alpha^+ = -\frac{D}{\omega_0} = -\sqrt{\frac{2}{\omega_0}} \frac{\partial}{\partial T_s} + \frac{\omega_0}{2 \sqrt{\gamma}} T_s,
\]
where \( \alpha \) and \( \alpha^+ \) satisfy the commutation relation \([\alpha, \alpha^+] = 1\), then
\[
\tilde{L}_k = -\gamma b^+ b - \omega_0 (ab^+ - a^+ b).
\]

As is known from its steady state solution, \( \lambda = 0, \tilde{\varphi}_{0,0} \) can be obtained from the following equations
\[
\alpha \tilde{\varphi}_{0,0} = b \tilde{\varphi}_{0,0} = 0,
\]
\[
\tilde{\varphi}_{0,0}(T_s, v) = \sqrt{\frac{\omega_0}{2\pi q}} \exp(-\frac{\gamma}{4q} v^2 + \frac{\omega_0^2 T_s^2}{q}).
\]
and its corresponding probability distribution is

\[
W(T_s, v) = |\varphi_{0,0}(T_s, v)|^2 = \frac{\omega_0^2}{2\pi^2} \exp\left[-\frac{\omega_0^2}{2\pi^2}(v^2 + \omega_0^2 T_s^2)\right],
\]

(32)

which is in agreement with that of steady state solution. The value of \(\langle T_s \rangle\) is calculated from Eq.(12) by using the values of parameter \(\alpha_1, \alpha_2, \beta_1, \beta_2,\) and \(q\) in the paper by Li and Huang (1984),

\[
\langle T_s \rangle = \int T_s W(T_s, v) dT_s dv = 2.71^\circ C,
\]

(33)

and this result is consistent with the range of 0.5-3.5\(^\circ\)C of sea surface temperature anomalies.

In order to write Eq.(30) into a simpler form, a set of new Boson-operators are introduced,

\[
C^+_1 = \delta^{-\frac{1}{2}}(\sqrt{\lambda_1}b^+ - \sqrt{\lambda_2}\alpha^+),
\]

\[
C^-_1 = \delta^{\frac{1}{2}}(\sqrt{\lambda_1}b^+ + \sqrt{\lambda_2}\alpha^+),
\]

\[
C^+_2 = \delta^{-\frac{1}{2}}(\sqrt{\lambda_1}b^+ - \sqrt{\lambda_2}b^-),
\]

\[
C^-_2 = \delta^{\frac{1}{2}}(\sqrt{\lambda_2}b^- + \sqrt{\lambda_1}\alpha^-),
\]

(34)

where \(\lambda_1\) and \(\lambda_2\) are the eigenvalues of the equation when no noise exists (see Rishen, 1984),

\[
\lambda_1 = (\gamma + \delta)/2, \quad \lambda_2 = (\gamma - \delta)/2,
\]

(35)

\[
\delta = \sqrt{\gamma^2 - 4\omega_0^2} = \lambda_1 - \lambda_2.
\]

(36)

Thus, \(\hat{L}_k\) becomes the sum of two simple harmonic vibrator Hamilton operators:

\[
\hat{L}_k = -\lambda_1 C^+_1 C^-_1 - \lambda_2 C^+_2 C^-_2.
\]

(37)

The commutation relations of \(C^+_1, C^-_1, C^+_2, \) and \(C^-_2\) are as follows

\[
\begin{align*}
[C^+_1, C^-_1] &= [C^+_2, C^-_2] = 1, \\
[C^-_1, C^+_2] &= [C^-_2, C^+_1] = [C^-_1, C^+_1] = [C^-_2, C^+_2] = 0, \\
[(-\hat{L}_k), C^+_i] &= \pm \lambda_i C^+_i \quad (i = 1, 2).
\end{align*}
\]

(38)

As is known from quantum mechanics, the variable \(\lambda_{n_1, n_2}\), and corresponding eigenfunctions can be obtained

\[
\varphi_{n_1, n_2}(T_s, v) = (n_1! n_2!)^{-\frac{1}{2}} (C^+_1)^{n_1} (C^-_2)^{n_2} \varphi_{0,0}(T_s, v),
\]

(39)

\[
\lambda_{n_1, n_2} = \lambda_1 n_1 + \lambda_2 n_2
\]

(40)

\[
\varphi^*_{n_1, n_2}(T_s, v) = (n_1!)^{-\frac{1}{2}} (n_2!)^{-\frac{1}{2}} (C^+_1)^{n_1} (C^-_2)^{n_2} \varphi^*_{0,0}(T_s, v),
\]

(41)

\[
W_\lambda(T_s, v) = |\varphi^*_{n_1, n_2} \cdot \varphi_{n_1, n_2}|,
\]

(42)

where \(n_1\) and \(n_2\) are positive integers. The probability distribution function \(W_\lambda\) expressed in terms of eigenfunction \(\varphi\) can be obtained from Eqs.(39)-(42), and the mean temperature anomaly \(\langle T_s \rangle\) can be calculated from Eq.(12). Equations (37)-(42) are obtained not under the condition of \(\partial W/\partial t = 0\), therefore \(W\) and \(T_s\) are both time-dependent solutions.

4. Results and discussion

(1) If \(N = n_1 + n_2 = 0\), i.e., \(n_1 = 0, n_2 = 0\), this is right a steady state solution, i.e., the ground state, and it represents a Brownian movement. Therefore, the point of view of Hasselmann’s stochastic climatic model, i.e., considering climatic changes as a Brownian movement of light and heavy particles collision (see Hasselmann, 1976), is theoretically proved in this paper. Viewed from the angle of quantum mechanics, the possible behavior of oceans is mostly in the ground state, and the sea temperature anomaly oscillates around its mean value 2.71\(^\circ\)C within a range of 0.5-3.5\(^\circ\)C. Disturbed by factors such as sunspots, volcanic eruption, polar ice changes etc., the air-sea system may transit from the ground state to first excited state or second excited state or even higher order excited state. However, because the distribution function is positively proportional to \(\exp(-\lambda t)\), the higher the order of excited state, the shorter the state lasts. Considering that the higher order excited state has less opportunity to occur and decays rapidly, we only discuss the first excited state here.

(2) If \(N = n_1 + n_2 = 1\), i.e., \(n_1 = 0, n_2 = 1\) or \(n_1 = 1, n_2 = 0\), the eigenvalues of system are

\[
\lambda_{0,1} = \frac{1}{2}(\gamma - \delta) = \lambda_1, \quad \lambda_{1,0} = \frac{1}{2}(\gamma + \delta) = \lambda_2.
\]

(43)
therefore, $\lambda_1, \lambda_2$ are only the particular cases of $\lambda_{n_1, n_2}$.

When and only when $\gamma >> 2\omega_0$, namely, only when the self-feedback coefficient is permanently greater than the mutual-feedback one, can $\lambda_{n_1, n_2}$ be reduced to

$$\lambda_{n_1, n_2} = \gamma n_1 + (\omega_0^2/\gamma)(n_2 - n_1) + O(\gamma^{-3}).$$

(44)

It is known from Eq. (36),

$$\delta = (\gamma^2 - 4\omega_0^2)^{1/2} \approx \gamma,$$

(45)

therefore,

$$\lambda_{0,1} = \lambda_1 \approx 0, \quad \lambda_{1,0} = \lambda_2 \approx \gamma.$$  

(46)

The eigenfunctions of first excited state are

$$\varphi_{1,0}(T_s, v, t) = (C^+) \varphi_{0,0}(T_s, v)e^{-\lambda_2 t},$$

$$= \delta^{-1/2} \sqrt{\lambda_2} \varphi_{0,0}(T_s, v)e^{-\lambda_2 t},$$

(47)

$$\overline{\varphi}_{1,0}(T_s, v, t) = \delta^{-1/2} \sqrt{\lambda_2} \varphi_{0,0}(T_s, v)e^{-\lambda_2 t},$$

(48)

then, the possible probability distribution $W(T_s, v, t)$ of first excited state is

$$W(T_s, v, t) = |\varphi_{1,0}(T_s, v, t)|^2$$

$$= \varphi_{1,0}^*(T_s, v, t)\varphi_{1,0}(T_s, v, t)$$

$$= A \cdot \exp\left[-\frac{\gamma}{2\delta^2}(v^2 + \omega_0^2 T_s^2)\right] T_s^2 e^{-2\lambda_2 t},$$

(49)

and

$$W(T_s, t) = \int W(T_s, v, t)dv$$

$$= B \cdot \exp\left(\frac{\gamma}{2\delta^2} \omega_0^2 T_s^2\right) T_s^2 e^{-2\lambda_2 t},$$

(50)

where $A$ and $B$ are normalizing coefficients. Equations (49) and (50) essentially differ from that of the steady ground state in the time decaying factor of $\exp(-\lambda t)$. Based on Eqs. (12) and (50), the mathematic expectation of sea temperature anomaly $T_s$ is

$$\langle T_s \rangle = \int_0^{\infty} W(T_s, t)dt/T_s/\int_0^{\infty} W(T_s, t)dt$$

$$= \frac{2}{\sqrt{2q\omega_0^2\pi}} e^{-2\lambda_2 t}.$$  

(51)

Set the carbon dioxide concentration to be $330 \times 10^{-6}$, the calculated $\alpha_1, \alpha_2, \beta_1, \beta_2$, and $q$ are 0.0001, 0.5127, 0.2186, -0.03921 rad month$^{-1}$, and 1.5 rad yr$^{-1}$, respectively: when the carbon dioxide concentration is doubled, calculated $\alpha_1, \alpha_2, \beta_1, \beta_2$, and $q$ become 0.0982, 0.5126, 0.2186, 0.03919, and 1.5 rad month$^{-1}$, respectively, then

$$\langle T_s \rangle_{330} = 0.068e^{-0.96t},$$

(52)

$$\langle T_s \rangle_{660} = 0.192e^{-0.71t}.$$  

(53)

When the carbon dioxide concentration is doubled, and under the condition of first excited state, the increment in the mean value of sea temperature anomaly calculated from Eqs. (52)-(53) is about 0.12°C; while under the condition of ground state, the corresponding increment is 1.81°C. However, the interesting thing is 0.71 and 0.96 in the exponential term. That is to say, when the carbon dioxide concentration is doubled, the decay becomes slow. This is a meaningful result. As shown in Figs.1 and 2, the doubling of CO$_2$ not only makes oceans warm, but also extends its decay period, thus indicating that oceans have a bigger thermal inertia. This result accords with physical consideration.

Delay time is generally used to compare the extent of decay. Since $\tau_{330}=1.096=1.04$ (month), $\tau_{660}=1/0.71=1.41$ (month), then $\tau_{660} > \tau_{330}$. When the CO$_2$ concentration is 330 ppm, it is 1.04 months for $\langle T_s \rangle$ to decay to 37% of itself; but when the CO$_2$ concentration is 660 ppm, it is 1.41 months. As known...
Fig. 2. As in Fig. 1, but for the doubling CO$_2$ concentration (660 ppm).

from Eqs. (35) and (40), when $\gamma < 2\omega_0$, then $\lambda_{n_1,n_2}$ is a complex number.

$$\delta = i \sqrt{4\omega_0^2 - \gamma^2} = 2i\omega,$$

$$\lambda_{n_1,n_2} = \frac{1}{2}(\gamma(n_1 + n_2) + i\omega(n_1 - n_2)), \quad (54)$$

thus, the probability distribution function puts up periodic oscillations with its period being $\frac{2\pi}{2\omega} = \frac{\pi}{\omega}$. The eigenvalues of first excited state are

$$\lambda_{1,0} = \frac{1}{2}(\gamma + i2\omega), \quad \lambda_{0,1} = \frac{1}{2}(\gamma - i2\omega), \quad (55)$$

and their oscillation forms are in agreement, only the form of $\lambda_{1,0} = \frac{1}{2}(\gamma + i2\omega)$ is discussed. Its eigenfunctions are

$$\phi_{1,0}(T_s, v, t) = \delta^{-\frac{1}{2}}(\sqrt{\lambda_1 b^+} - \sqrt{\lambda_2 a^+})\phi_{0,0}(T_s, v) \approx (Av + BT_s)e^{-\frac{\omega}{2}\left(v^2 + \omega_0^2T_s^2\right)}e^{-\omega t}e^{-i\omega t}, \quad (56)$$

$$\phi_{1,0}^*(T_s, v, t) \approx (Av + BT_s)e^{-\frac{\omega}{2}\left(v^2 + \omega_0^2T_s^2\right)}e^{-\lambda_{1,0}t}, \quad (57)$$

where $A$ and $B$ are coefficients, and determined by normalizing condition. The following equations can be obtained from Eqs. (49)-(51):

$$\langle T_{s\rangle 330} = 0.97e^{-0.5127t}e^{-i0.2243t}, \quad (58)$$

$$\langle T_{s\rangle 660} = 1.82e^{-0.5126t}e^{-i0.2153t}, \quad (59)$$

It can be seen from Eqs. (58)-(59) that when the CO$_2$ concentration is doubled, the increment of temperature is $0.85^\circ$C. The temporal variation in the real part of $\langle T_{s\rangle 330}$ is close to that of $\langle T_{s\rangle 660}$; while the period $T_{330} = \frac{2\pi}{0.2243} = 2.3$ yr and $T_{660} = \frac{2\pi}{0.2153} = 2.4$ yr, as shown in Figs. 3 and 4. This result can be used to explain the quasi-biennial periodic oscillations in the atmosphere and oceans.

It can be clearly seen from Figs. 3 and 4 that when
the CO₂ concentration is doubled, the oscillation period is prolonged, although not so much. This indicates that under the influence of the increment of temperature induced by CO₂-doubling, the periodic response time of the air-sea coupling system to external disturbances will be prolonged. This assumption remains to be confirmed by other simulation results.

REFERENCES